

Axiomatizing Mathematical Theories: Multiplication*

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Abstract

Axiomatizing mathematical structures is a goal of Mathematical Logic. Axiomatizability of the theories of some structures have turned out to be quite difficult and challenging, and some remain open. However axiomatization of some mathematical structures are now classical theorems in Logic, Algebra and Geometry. In this paper we will study the axiomatizability of the theories of multiplication in the domains of natural, integer, rational, real, and complex numbers. We will review some classical theorems, and will give some new proofs for old results. We will see that some structures are missing in the literature, thus leaving it open whether the theories of that structures are axiomatizable (decidable) or not. We will answer one of those open questions in this paper.

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1 Introduction

Dedicated to Professor SIAVASH SHAHSHAHANI for his 70th Birthday.

Classic Algebraic Geometry is about studying properties of the sets of zeros of polynomials; the word geometry refers to the set of zeros of (a system of) polynomials. Studying the systems of linear equations with one variable is a subject of Linea Algebra, and studying polynomial equations in one variable is a subject of Algebra. In Algebraic Geometry these two areas are combined by studying systems of polynomials in several variables. Chevalley–Tarski Theorem in algebraic geometry states that the projection of a constructible set is constructible; in mathematical logic this theorem implies that the theory of algebraically closed fields, or equivalently the theory of the structure $\langle \mathbb{C}, 0, 1, +, -, \cdot \rangle$, admits quantifier elimination. As a result the theory of the structure $\langle \mathbb{C}, +, \cdot \rangle$ is decidable and can be axiomatized as an *algebraically closed field*. A set D is called decidable when there exists an algorithm which for a given input x outputs **yes** if $x \in D$ and outputs **no** if $x \notin D$. Tarski–Seidenberg Theorem (or Tarski–Seidenberg Principle) in algebraic geometry states that the projection of a semialgebraic set is a semialgebraic set; in mathematical logic this theorem implies that the theory of the real closed (ordered) fields, or equivalently the theory of the structure $\langle \mathbb{R}, 0, 1, +, -, \cdot, < \rangle$, admits quantifier elimination. As a result the theory of the structure $\langle \mathbb{R}, +, \cdot \rangle$ is decidable and can be axiomatized as a *real closed (ordered) field*. Let us note that ordering can be defined by addition and multiplication in \mathbb{R} : for real a, b we have $a \leq b$ if and only if $\exists x(a + x^2 = b)$.

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One of the most surprising and most significant results in the twentieth century mathematics was Gödel's Incompleteness Theorem. A semantical reading of this theorem is that the the additive and multiplicative theory of the natural numbers, $\langle \mathbb{N}, +, \cdot \rangle$, is not decidable. Undecidability of the theory of the structure $\langle \mathbb{N}, +, \cdot \rangle$ means that there is no algorithm which for a given first order sentence φ in the language $\{+, \cdot\}$ (as input) can decide (output **yes**) if $\mathbb{N} \models \varphi$ or not (output **no** if $\mathbb{N} \not\models \varphi$). The theory of $\langle \mathbb{Z}, +, \cdot \rangle$ is not decidable either because \mathbb{N} is definable in it, and also the theory of $\langle \mathbb{Q}, +, \cdot \rangle$ is undecidable since \mathbb{Z} is definable in it.

One surprising aspect of Gödel's 1931 incompleteness theorem (undecidability of the theory $\langle \mathbb{N}, +, \cdot \rangle$) was that by the results of Presburger and Skolem the additive theory of the natural numbers without multiplication, $\langle \mathbb{N}, + \rangle$, and the multiplicative theory of the natural numbers without addition, $\langle \mathbb{N}, \cdot \rangle$, were proved (or announced) to be decidable by 1930. So, the theory of the structures $\langle \mathbb{N}, +, \cdot \rangle$ was expected to be decidable, which turned out not to be so by Kurt Gödel. Indeed the theories of the structures $\langle \mathbb{Z}, + \rangle$ and $\langle \mathbb{Z}, \cdot \rangle$ are also decidable, and the theories of the structures $\langle \mathbb{R}, + \rangle$, $\langle \mathbb{R}, \cdot \rangle$, $\langle \mathbb{C}, + \rangle$ and $\langle \mathbb{C}, \cdot \rangle$ are evidently decidable by the above mentioned theorems of Tarski (and Chevalley and Seidenberg). Our aim is to study the following structures of multiplication in the domains of \mathbb{N} , \mathbb{Z} , \mathbb{Q} , \mathbb{R} and \mathbb{C} .

	\mathbb{N}	\mathbb{Z}	\mathbb{Q}	\mathbb{R}	\mathbb{C}
$\{\cdot\}$	$\langle \mathbb{N}, \cdot \rangle$	$\langle \mathbb{Z}, \cdot \rangle$	$\langle \mathbb{Q}, \cdot \rangle$	$\langle \mathbb{R}, \cdot \rangle$	$\langle \mathbb{C}, \cdot \rangle$
$\{+, \cdot\}$	$\langle \mathbb{N}, +, \cdot \rangle$	$\langle \mathbb{Z}, +, \cdot \rangle$	$\langle \mathbb{Q}, +, \cdot \rangle$	$\langle \mathbb{R}, +, \cdot \rangle$	$\langle \mathbb{C}, +, \cdot \rangle$

Though the theory of the structures $\langle \mathbb{R}, \cdot \rangle$ and $\langle \mathbb{C}, \cdot \rangle$ are decidable by Tarski's theorem (decidability of the theories of $\langle \mathbb{R}, +, \cdot \rangle$ and $\langle \mathbb{C}, +, \cdot \rangle$), we will prove this fact directly (without appealing to Tarski's results) by quantifier elimination. Here, the decidability or undecidability of the structure $\langle \mathbb{Q}, \cdot \rangle$ seems to be missing in the literature. The theory $\langle \mathbb{Q}, +, \cdot \rangle$ is not decidable, so one cannot immediately infer the decidability of the theory of $\langle \mathbb{Q}, \cdot \rangle$. In this paper we show the decidability of the theory of this structure by the method of quantifier elimination. Whence, we will give some nice characterizations for the theory of multiplication in the fields of rational, real, and complex numbers. The status of decidability of the theories of the above mentioned structures has been summarized in the below table, where decidable theories are indicated by Δ_1 and undecidable theories by \nexists_1 .

	\mathbb{N}	\mathbb{Z}	\mathbb{Q}	\mathbb{R}	\mathbb{C}
$\{\cdot\}$	Δ_1	Δ_1	Δ_1	Δ_1	Δ_1
$\{+, \cdot\}$	\nexists_1	\nexists_1	\nexists_1	Δ_1	Δ_1

2 The Theory of Multiplication

2.1 Natural and Integer Numbers

There are some model-theoretic (and advanced) proofs for the decidability of the theory of $\langle \mathbb{N}, \cdot \rangle$ in the literature, the most elementary (and in the author's opinion, the most interesting) proof is the one given by Patrick Cegielski in [3] which uses the technique of quantifier elimination. Unfortunately, there is no English version of this proof (see [8] for an overview of it). As far as the author knows, there is no direct reference to the decidability of the theory of the structure $\langle \mathbb{Z}, \cdot \rangle$ in the literature, but somehow it seems that Cegielski's quantifier elimination proof can also be adopted to the case of \mathbb{Z} . Thus the multiplicative theory of \mathbb{Z} is most likely decidable, and most likely can be proved by the method of quantifier elimination.

2.2 Rational Numbers

Let us note that the language $\{\cdot\}$ does not allow quantifier elimination for $\langle \mathbb{Q}, \cdot \rangle$, since e.g. the formula $\exists y[x = y^2]$ is not equivalent to a quantifier-free formula. So, we first restrict our attention to the set of positive rational numbers $\mathbb{Q}^+ = \{r \in \mathbb{Q} \mid r > 0\}$ and extend the language to $\mathcal{L} = \langle 1, \cdot, {}^{-1}, R_2, R_3, \dots \rangle$, where R_n is interpreted as “being the n th power of a rational”; or in the other words $R_n(x) \equiv \exists y[x = y^n]$. We note that the quantifier-free formulas of \mathcal{L} are decidable: for any given rational number r and any natural n one can decide if r is an n th power of (an-)other rational number or not. Thus, quantifier elimination in $\langle \mathbb{Q}^+, \mathcal{L} \rangle$ implies the decidability of the theory of the structure $\langle \mathbb{Q}^+, \mathcal{L} \rangle$, and hence decidability of the theory of $\langle \mathbb{Q}^+, \cdot \rangle$. We will need the following general form of the Chinese Remainder Theorem ([6]).

Theorem 1 (General Chinese Remainder) *The system $\{x \equiv_{n_i} u_i\}_{i=1}^l$ of congruence equations has a solution in \mathbb{Z} if and only if for every $i \neq j$, $u_i \equiv_{(n_i, n_j)} u_j$ where (n_i, n_j) is the greatest common divisor of n_i and n_j . Moreover the solution x_0 (if exists) is unique module $n = [n_1, \dots, n_l]$ (the least common multiple of n_i ’s), and so all the solutions will be of the form $N \cdot n + x_0$ for arbitrary $N \in \mathbb{Z}$.*

Proof. If x_0 satisfies $\bigwedge_i x_0 \equiv_{n_i} u_i$ then for any $i \neq j$ we will have $x_0 \equiv_{(n_i, n_j)} u_i$ and $x_0 \equiv_{(n_i, n_j)} u_j$, whence $u_i \equiv_{(n_i, n_j)} u_j$. Conversely, if $\bigwedge_{i \neq j} u_i \equiv_{(n_i, n_j)} u_j$ then we show the existence of some x_0 which satisfies $\bigwedge_i x_0 \equiv_{n_i} u_i$. Then of course every number $x = N \cdot n + x_0$ satisfies the system of equations $\bigwedge_i x \equiv_{n_i} u_i$ as well, and every solution y of the equations $\bigwedge_i y \equiv_{n_i} u_i$ satisfies $\bigwedge_i x_0 \equiv_{n_i} y$, and so $x_0 \equiv_n y$, therefore $y = N \cdot n + x_0$ for some $N \in \mathbb{N}$. Since the greatest common divisor of the numbers $\{n/n_1, \dots, n/n_l\}$ is 1, there are c_1, \dots, c_l such that $\sum_i c_i(n/n_i) = 1$. Since for any $i \neq j$, $(n_i, n_j) \mid u_i - u_j$ and $[n_i, n_j] \mid [n_1, \dots, n_l] = n$ there are $d_{i,j}$ and $e_{i,j}$ such that $u_i - u_j = d_{i,j}(n_i, n_j)$ and $n = e_{i,j}[n_i, n_j]$. Whence, by $(n_i, n_j)[n_i, n_j] = n_i n_j$ we have $(u_i - u_j)n/n_i = d_{i,j} \cdot (n_i, n_j) \cdot e_{i,j} \cdot [n_i, n_j]/n_i = d_{i,j} \cdot e_{i,j} \cdot n_j$. Put $x_0 = \sum_i u_i \cdot c_i \cdot (n/n_i)$. Then

$$\begin{aligned} x_0 &= u_j c_j n/n_j + \sum_{i \neq j} u_i \cdot c_i \cdot (n/n_i) \\ &= u_j c_j n/n_j + \sum_{i \neq j} (u_i - u_j) \cdot c_i \cdot (n/n_i) + \sum_{i \neq j} u_j \cdot c_i \cdot (n/n_i) \\ &= u_j \cdot \sum_i c_i \cdot (n/n_i) + \sum_{i \neq j} (u_i - u_j) \cdot c_i \cdot (n/n_i) \\ &= u_j + \sum_{i \neq j} c_i \cdot (u_i - u_j) \cdot (n/n_i) \\ &= u_j + \sum_{i \neq j} c_i \cdot d_{i,j} \cdot e_{i,j} \cdot n_j \\ &= u_j + n_j \cdot \sum_{i \neq j} c_i \cdot d_{i,j} \cdot e_{i,j}, \end{aligned}$$

which implies the desired conclusion $x_0 \equiv_{n_j} u_j$ (for every $j = 1, \dots, l$). \square

Lemma 2 *The system of relations $\{R_{n_i}(u_i \cdot x)\}_{i=1}^l$ (recall that $R_n(x) \equiv \exists y[x = y^n]$) has a solution in \mathbb{Q}^+ if and only if for every $i \neq j$, $R_{(n_i, n_j)}(u_i \cdot u_j^{-1})$ holds where (n_i, n_j) is the greatest common divisor of n_i and n_j . Moreover if $\bigwedge_{i \neq j} R_{(n_i, n_j)}(u_i \cdot u_j^{-1})$ holds then for $n = [n_1, \dots, n_l]$ and for some fixed $c_1, \dots, c_l \in \mathbb{Z}$ which satisfy $\sum_i c_i \cdot n/n_i = 1$ all of the solutions are of the form $w^n \cdot \prod_{i=1}^l (u_i)^{-c_i \cdot n/n_i}$ for arbitrary $w \in \mathbb{Q}$.*

Proof. Clearly, if $R_{n_i}(u_i \cdot x)$ and $R_{n_j}(u_j \cdot x)$ then $R_{(n_i, n_j)}(u_i \cdot x)$ and $R_{(n_i, n_j)}(u_j^{-1} \cdot x^{-1})$, and so $R_{(n_i, n_j)}(u_i \cdot u_j^{-1})$. Conversely, if $\bigwedge_{i \neq j} R_{(n_i, n_j)}(u_i \cdot u_j^{-1})$ holds and $n = [n_1, \dots, n_l]$ then since the greatest common divisor of n/n_i ’s is 1 there are some c_1, \dots, c_l such that $\sum_i c_i \cdot n/n_i = 1$.

We show that $x_0 = \prod_{i=1}^l (u_i)^{-c_i \cdot n/n_i}$ satisfies $\bigwedge_i R_{n_i}(u_i \cdot x_0)$. Note that every rational number can be uniquely factorized into a product of some (powers of) primes $p_1^{\beta_1} p_2^{\beta_2} \cdots p_m^{\beta_m}$ where $\beta_1, \beta_2, \dots, \beta_m \in \mathbb{Z}$ (for example $24/45 = 2^3 3^{-1} 5^{-1}$). For a fix prime \mathfrak{p} , assume the exponents of \mathfrak{p} in the unique factorizations of u_1, \dots, u_l are respectively $\alpha_1, \dots, \alpha_l$. Then the exponent of \mathfrak{p} in the unique factorization of x_0 will be $\alpha = \sum_i -c_i \alpha_i (n/n_i)$. Also, by the assumption $\bigwedge_{i \neq j} R_{(n_i, n_j)}(u_i \cdot u_j^{-1})$ we have $\bigwedge_{i \neq j} \alpha_i \equiv_{(n_i, n_j)} \alpha_j$. So, by the proof of the General Chinese Remainder Theorem 1, $\bigwedge_i \alpha \equiv_{n_i} -\alpha_i$, or $\bigwedge_i n_i \mid \alpha_i + \alpha$. This means that the exponent of (every prime) in the unique factorization of $u_i \cdot x_0$ is a multiple of n_i , whence $R_{n_i}(u_i \cdot x_0)$ holds (for $i = 1, \dots, l$). Now assume for $y \in \mathbb{Q}$ the relation $\bigwedge_i R_{n_i}(u_i \cdot y)$ holds. Then for any prime \mathfrak{p} , if the exponent of \mathfrak{p} in the unique factorization of y is β , we have $\bigwedge_i \beta \equiv_{n_i} -\alpha_i$. Whence, by the proof of Theorem 1 we have $\beta \equiv_n \alpha$, and so $y = w^n \cdot x_0$ holds for some $w \in \mathbb{Q}$. \square

Lemma 3 *The system of relations $\{R_{n_j}(u_j \cdot x)\}_{j=1}^l, \{\neg R_{m_k}(v_k \cdot x)\}_{k=1}^\ell$ has a solution in \mathbb{Q}^+ if and only if $\bigwedge_{i \neq j} R_{(n_i, n_j)}(u_i \cdot u_j^{-1}) \wedge \bigwedge_{k: m_k \mid n} \neg R_{m_k}(v_k \cdot x_0)$ holds where $n = [n_1, \dots, n_l]$ and $x_0 = \prod_{i=1}^l (u_i)^{-c_i \cdot n/n_i}$ in which c_i 's satisfy $\sum_i c_i \cdot n/n_i = 1$ (by $(n/n_1, \dots, n/n_l) = 1$).*

Proof. Suppose the system $\{R_{n_j}(u_j \cdot x)\}_{j=1}^l, \{\neg R_{m_k}(v_k \cdot x)\}_{k=1}^\ell$ has a solution (for x) in \mathbb{Q}^+ . Then by Lemma 2, $\bigwedge_{i \neq j} R_{(n_i, n_j)}(u_i \cdot u_j^{-1})$ holds, and moreover x is of the form $w^n \cdot x_0$ for some $w \in \mathbb{Q}^+$. We show that $\bigwedge_{k: m_k \mid n} \neg R_{m_k}(v_k \cdot x_0)$ holds too. Suppose $m_k \mid n$. Then $v_k \cdot x = v_k \cdot w^n \cdot x_0$, and so by $R_{m_k}(w^n)$ and $\neg R_{m_k}(v_k \cdot x)$ we infer that $\neg R_{m_k}(v_k \cdot x_0)$. Conversely, suppose

$$\bigwedge_{i \neq j} R_{(n_i, n_j)}(u_i \cdot u_j^{-1}) \wedge \bigwedge_{k: m_k \mid n} \neg R_{m_k}(v_k \cdot x_0).$$

Then by Lemma 2 for any $w \in \mathbb{Q}^+$ the number $x = w^n \cdot x_0$ satisfies $\bigwedge_j R_{n_j}(u_j \cdot x)$. We choose a suitable w for which $x = w^n \cdot x_0$ also satisfies $\bigwedge_k \neg R_{m_k}(v_k \cdot x)$. Choose P a prime number which does not appear in the (unique) factorization of any of u_1 or u_2 or ... or u_l or v_1 or ... or v_ℓ . Now we show that $x = P^n \cdot x_0$ satisfies $\bigwedge_k \neg R_{m_k}(v_k \cdot x)$:

- (i) If $m_k \mid n$ then $\neg R_{m_k}(v_k \cdot x_0)$ and $R_{m_k}(P^n)$; whence $\neg R_{m_k}(v_k \cdot x)$.
- (ii) If $m_k \nmid n$, then $\neg R_{m_k}(P^n)$ and so $\neg R_{m_k}(v_k \cdot x)$, because the prime number P does not appear in the unique factorization of x_0 or v_k (if we had $R_{m_k}(v_k \cdot x) \equiv R_{m_k}(v_k \cdot x_0 \cdot P^n)$ then we must have had $R_{m_k}(P^n)$ or $m_k \mid n$). \square

Theorem 4 *The theory of the structure $\langle \mathbb{Q}^+, 1, \cdot, ^{-1}, R_2, R_3, \dots \rangle$ admits quantifier elimination.*

Proof. The folklore technique of quantifier elimination starts from characterizing the terms and atomic formulas, also eliminating negations, implications and universal quantifiers, and then removing the disjunctions from the scopes of existential quantifiers, which leaves the final case to be the existential quantifier with the conjunction of some atomic (or negated atomic) formulas (see Theorem 31F of [4]). Removing this one existential quantifier implies the ability to eliminate all the other quantifiers by induction.

Let us summarize the first steps: For a variable x and parameter a , all \mathcal{L} -terms are equal to $x^k a^l$ for some $k, l \in \mathbb{Z}$. Atomic \mathcal{L} -formulas are in the form $s = t$ or $R_n(u)$ for some terms s, t, u and $n \geq 2$. Negated atomic \mathcal{L} -formulas are thus $s \neq t$ and $\neg R_n(u)$. So, we will eliminate the quantifier of the formulas of the form $\exists x (\bigwedge_i \theta_i)$ where each θ_i is in the form $(x^\alpha = s)$ or $(x^\beta \neq t)$ or $R_n(ux^\gamma)$ or $\neg R_m(vx^\delta)$ for some $\alpha, \beta, \gamma, \delta \in \mathbb{N}$ and \mathcal{L} -terms s, t, u, v . Whence, it

suffices to show that the \mathcal{L} -formula

$$\exists x [\bigwedge_h (x^{\alpha_h} = s_h) \wedge \bigwedge_i (x^{\beta_i} \neq t_i) \wedge \bigwedge_j (R_{n_j}(u_j \cdot x^{\gamma_j})) \wedge \bigwedge_k (\neg R_{m_k}(v_k \cdot x^{\delta_k}))]$$

is equivalent to another \mathcal{L} -formula in which $\exists x$ does not appear. This will finish the proof.

Here comes the next steps of quantifier elimination. The powers of x can be unified: let p be the least common multiple of the α_h 's, β_i 's, γ_j 's, and δ_k 's. From the $\langle \mathbb{Q}^+, \mathcal{L} \rangle$ -equivalences $a = b \leftrightarrow a^q = b^q$ and $R_n(a) \leftrightarrow R_{nq}(a^q)$ we infer that the above formula can be re-written equivalently as $\exists x [\bigwedge_h (x^p = s_h) \wedge \bigwedge_i (x^p \neq t_i) \wedge \bigwedge_j (R_{n_j}(u_j \cdot x^p)) \wedge \bigwedge_k (\neg R_{m_k}(v_k \cdot x^p))]$ for possibly new s_h 's, t_i 's, u_j 's, n_j 's, v_k 's, and m_k 's. This formula is in turn equivalent to

$$\exists y [\bigwedge_h (y = s_h) \wedge \bigwedge_i (y \neq t_i) \wedge \bigwedge_j (R_{n_j}(u_j \cdot y)) \wedge \bigwedge_k (\neg R_{m_k}(v_k \cdot y)) \wedge R_p(y)]$$

(with the substitution $y = x^p$). Thus it suffices to show that the following formula is equivalent to a quantifier-free formula: $\exists x [\bigwedge_h (x = s_h) \wedge \bigwedge_i (x \neq t_i) \wedge \bigwedge_j (R_{n_j}(u_j \cdot x)) \wedge \bigwedge_k (\neg R_{m_k}(v_k \cdot x))]$. If the conjunction $\bigwedge_h (x = s_h)$ is not empty ($h \neq 0$), then the above formula is equivalent to the quantifier-free formula (for some term s_0):

$$\bigwedge_h (s_0 = s_h) \wedge \bigwedge_i (s_0 \neq t_i) \wedge \bigwedge_j (R_{n_j}(u_j \cdot s_0)) \wedge \bigwedge_k (\neg R_{m_k}(v_k \cdot s_0)).$$

So, let us assume that the conjunction $\bigwedge_h (x = s_h)$ is empty ($h = 0$), and thus we are to eliminate the quantifier of the formula

$$(A) \quad \exists x [\bigwedge_i (x \neq t_i) \wedge \bigwedge_j (R_{n_j}(u_j \cdot x)) \wedge \bigwedge_k (\neg R_{m_k}(v_k \cdot x))].$$

By Lemma 3 this formula implies the following quantifier-free formula

$$(B) \quad \bigwedge_{i \neq j} R_{(n_i, n_j)}(u_i \cdot u_j^{-1}) \wedge \bigwedge_{k: m_k | n} \neg R_{m_k}(v_k \cdot x_0)$$

where $n = [n_1, \dots, n_l]$ and $x_0 = \prod_{i=1}^l (u_i)^{-c_i \cdot n / n_i}$ in which c_i 's satisfy $\sum_i c_i \cdot n / n_i = 1$. Also the proof of Lemma 3 shows that if (B) holds, then there are infinitely many x 's which satisfy $\bigwedge_j (R_{n_j}(u_j \cdot x)) \wedge \bigwedge_k (\neg R_{m_k}(v_k \cdot x))$, and so some of those x 's also satisfy $\bigwedge_i (x \neq t_i)$; whence (A) holds too. Summing up, (A) is equivalent to the quantifier-free formula (B). \square

The above proof can be adapted to show the quantifier-elimination for the theory of the structures $\langle \mathbb{Q}^{\geq 0}, 0, 1, \cdot, ^{-1}, R_2, R_3, \dots \rangle$ and $\langle \mathbb{Q}, 0, 1, -1, \cdot, ^{-1}, R_2, R_3, \dots, \mathcal{P} \rangle$, where the predicate \mathcal{P} is defined as $\mathcal{P}(x) \iff x > 0$. Since the techniques of the proofs of these theorems are very similar to the proofs in the next section, we do not present them.

Theorem 5 *Theories of the following structures admit quantifier elimination:*

$$\langle \mathbb{Q}^{\geq 0}, 0, 1, \cdot, ^{-1}, R_2, R_3, \dots \rangle \text{ and } \langle \mathbb{Q}, 0, 1, -1, \cdot, ^{-1}, R_2, R_3, \dots, \mathcal{P} \rangle.$$

Corollary 6 *The theories of the structures $\langle \mathbb{Q}^+, \cdot \rangle$, $\langle \mathbb{Q}^{\geq 0}, \cdot \rangle$, and $\langle \mathbb{Q}, \cdot \rangle$ are decidable.*

2.3 Real Numbers

The multiplicative theory of the positive real numbers, $\langle \mathbb{R}^+, \cdot \rangle$, can be axiomatized as Non-Trivial Torsion-Free Divisible Abelian Groups:

$$\begin{aligned} & \bullet \forall x, y, z (x \cdot (y \cdot z) = (x \cdot y) \cdot z) & \bullet \forall x (x \cdot 1 = x = 1 \cdot x) & \bullet \forall x (x \cdot x^{-1} = 1 = x^{-1} \cdot x) & \bullet \forall x, y (x \cdot y = y \cdot x) \\ & \bullet \forall x \exists y (y^n = x), \quad n = 2, 3, \dots & \bullet \forall x (x^n = 1 \rightarrow x = 1), \quad n = 2, 3, \dots & \bullet \exists x (x \neq 1) \end{aligned}$$

Theorem 7 *The theory of the structure $\langle \mathbb{R}^+, 1, \cdot, ^{-1} \rangle$ admits quantifier elimination.*

Proof. Every atomic formula containing the variable x can be written as $x^n = t$ for some $n \in \mathbb{Z}$ and some term t . So, it suffices to show that the formula

$$\exists x (\bigwedge_i x^{n_i} = t_i \wedge \bigwedge_j x^{m_j} \neq s_j)$$

is equivalent to a quantifier-free formula. Take p be the least common multiple of n_i 's and

m_j 's; then by $a = b \iff a^k = b^k$, the above formula will be equivalent to

$$\exists x \left(\bigwedge_i x^p = t_i \wedge \bigwedge_j x^p \neq s_j \right)$$

and by the divisibility of $\langle \mathbb{R}^+, 1, \cdot, {}^{-1} \rangle$ (the axiom $\forall x \exists y (y^n = x)$) we can write this as

$$\exists x \left(\bigwedge_i x = t_i \wedge \bigwedge_j x \neq s_j \right).$$

If the conjunct $\bigwedge_i x = t_i$ is not empty, then the above formula is equivalent to the following quantifier-free formula (for some term t_0):

$$\bigwedge_i t_0 = t_i \wedge \bigwedge_j t_0 \neq s_j,$$

and if the conjunct $\bigwedge_i x = t_i$ is empty, then the above formula (being $\exists x (\bigwedge_j x \neq s_j)$ which is true) is equivalent to $0 = 0$ which is a quantifier-free formula. \square

Theorem 8 *The theory of the structure $\langle \mathbb{R}^{\geq 0}, 0, 1, \cdot, {}^{-1} \rangle$ admits quantifier elimination.*

Proof. This can be proved just like Theorem 7, noting that $\forall x (x \cdot 0 = 0)$. \square

Let \mathcal{P} be the predicate of positiveness: $\mathcal{P}(x) \equiv x > 0$. Below we will show that the theory of the structure $\langle \mathbb{R}, 0, 1, -1, \cdot, {}^{-1}, \mathcal{P} \rangle$ admits quantifier elimination. By convention $x^{-1} = 1/x$ if $x \neq 0$ and $x^{-1} = 0$ if $x = 0$. The notation $-x$ is a shorthand for $(-1) \cdot x$.

Theorem 9 *The theory of the structure $\langle \mathbb{R}, 0, 1, -1, \cdot, {}^{-1}, \mathcal{P} \rangle$ admits quantifier elimination.*

Proof. All the atomic formulas of the language $\{0, 1, -1, \cdot, {}^{-1}, \mathcal{P}\}$ containing the variable x are in the form $x^n = t$ or $x^m \neq s$ or $\mathcal{P}(x^n t)$ or $\neg \mathcal{P}(x^m s)$ for some $n, m \in \mathbb{Z}$ and terms t, s . We note that negation can be eliminated from \mathcal{P} by $\neg \mathcal{P}(x) \iff x = 0 \vee \mathcal{P}(-x)$. Also by $\mathcal{P}(a \cdot b) \iff [\mathcal{P}(a) \wedge \mathcal{P}(b)] \vee [\mathcal{P}(-a) \wedge \mathcal{P}(-b)]$ and $\mathcal{P}(x^2)$ we can write $\mathcal{P}(x^n u)$ as $\mathcal{P}(u)$ if n is even, and as $[\mathcal{P}(x) \wedge \mathcal{P}(u)] \vee [\mathcal{P}(-x) \wedge \mathcal{P}(-u)]$ if n is odd. Thus for eliminating the quantifier of $\exists x (\bigwedge_i x^{n_i} = t_i \wedge \bigwedge_j x^{m_j} \neq s_j \wedge \bigwedge_k \mathcal{P}(x^{\alpha_k} u_k) \wedge \bigwedge_l \neg \mathcal{P}(x^{\beta_l} v_l))$ we need to consider the formulas of the form $\exists x (\bigwedge_i x^{n_i} = t_i \wedge \bigwedge_j x^{m_j} \neq s_j \wedge \mathcal{P}(x))$ or $\exists x (\bigwedge_i x^{n_i} = t_i \wedge \bigwedge_j x^{m_j} \neq s_j \wedge \neg \mathcal{P}(x))$ only. Note that the second formula is equivalent to the following formula (by $y = -x$):

$$\exists y (\bigwedge_i y^{n_i} = (-1)^{n_i} \cdot t_i \wedge \bigwedge_j y^{m_j} \neq (-1)^{m_j} s_j \wedge \mathcal{P}(y)).$$

So, we consider the following formulas for eliminating their quantifiers:

$$(A) \quad \exists x (\mathcal{P}(x) \wedge \bigwedge_i x^{n_i} = t_i \wedge \bigwedge_j x^{m_j} \neq s_j).$$

Note that when $\mathcal{P}(x)$ does not appear, then we can again write every formula $\exists x \varphi(x)$ as

$$\exists x (\mathcal{P}(x) \wedge \varphi(x)) \vee \varphi(0) \vee \exists x (\neg \mathcal{P}(x) \wedge \varphi(-x)).$$

For eliminating the quantifier of the formula (A) we distinguish the signs of the other variables appearing in it by the above trick. So, if y_1, \dots, y_ℓ are all the variables (other than x) appearing in (A), then (A) is equivalent to the following formula

$$(A') \quad \bigvee_{\pi \in 3^\ell} \exists x (\mathcal{P}(x) \wedge \bigwedge_{k=1}^\ell \mathcal{P}^{\pi(y_k)}(y_k) \wedge \bigwedge_i x^{n_i} = t_i \wedge \bigwedge_j x^{m_j} \neq s_j)$$

where $3^\ell = \{\pi \mid \pi : \{y_1, \dots, y_\ell\} \rightarrow \{-1, 0, 1\}\}$ and $\mathcal{P}^{\pi(y)}(y)$ is defined as $\mathcal{P}^{\pi(y)}(y) = \mathcal{P}(y)$ if $\pi(y) = 1$, $\mathcal{P}^{\pi(y)}(y) = "y = 0"$ if $\pi(y) = 0$, and $\mathcal{P}^{\pi(y)}(y) = \mathcal{P}(-y)$ if $\pi(y) = -1$. By changing y_i to $-y_i$ when necessary, then we need to eliminate the quantifier of the formula

$$(B) \quad \exists x (\mathcal{P}(x) \wedge \bigwedge_{k=1}^\ell \mathcal{P}(y_k) \wedge \bigwedge_i x^{n_i} = t_i(y_1, \dots, y_\ell) \wedge \bigwedge_j x^{m_j} \neq s_j(y_1, \dots, y_\ell))$$

where x and y_k 's are all the appearing variables. By the equivalences $\mathcal{P}(z) \wedge \mathcal{P}(-z) \leftrightarrow 0 = 1$ and $\mathcal{P}(u) \wedge \mathcal{P}(v) \rightarrow u \neq -v$, we can assume that no formula of the form $x^{n_i} = -y_1^{\alpha_1} \dots y_\ell^{\alpha_\ell}$ (which is false) or of the form $x^{m_j} \neq -y_1^{\alpha_1} \dots y_\ell^{\alpha_\ell}$ (which is true) appears in (B). Or in the other words, no -1 should appear in (B). Finally, Theorem 7 implies that the formula (B) is equivalent to

a quantifier-free formula, since x and all the y_k 's are supposed to be in \mathbb{R}^+ by the assumption $\mathcal{P}(x) \wedge \bigwedge_{k=1}^{\ell} \mathcal{P}(y_k)$. \square

Let us note that we did not use ordering of reals in the above proof, and in fact $<$ cannot be defined by multiplication in $\langle \mathbb{R}, \cdot \rangle$.

2.4 Complex Numbers

We will give a direct proof for the decidability of the theory of the structure $\langle \mathbb{C}, \cdot \rangle$ (without referring to Tarski's theorem about the decidability of the theory of algebraically closed fields) by showing the quantifier elimination of a suitable extension of the language $\{\cdot\}$. For any $n \geq 3$, let $\omega_n = \cos(2\pi/n) + i \sin(2\pi/n)$. The complex number ω_n is an n -th root of the unit, and indeed all the n -th roots of the unit are $\omega_n, (\omega_n)^2, \dots, (\omega_n)^n = 1$. That is to say that $\{z \in \mathbb{C} \mid z^n = 1\} = \{(\omega_n)^i \mid 1 \leq i \leq n\}$. Let us recall that by convention $0^{-1} = 0$.

Theorem 10 *The theory of $\langle \mathbb{C}, 0, 1, -1, \cdot, ^{-1}, \omega_3, \omega_4, \dots \rangle$ admits quantifier elimination.*

Proof. Just like the proof of Theorem 7 it suffices to show that the formula

$$(A) \quad \exists x \left(\bigwedge_i x^{n_i} = t_i \wedge \bigwedge_j x^{m_j} \neq s_j \right),$$

where t_i 's and s_j 's are multiplications of some variables or the constants $0, 1, -1, \omega_3, \omega_4, \dots$ or their inverses, is equivalent to a quantifier-free formula. We can assume that all n_i 's are equal, since if for example $n_i < n_j$ then the formula $\exists x (x^{n_i} = t_i \wedge x^{n_j} = t_j \wedge \theta)$ is equivalent to $\exists x (x^{n_i} = t_i \wedge x^{n_j - n_i} = t_j \cdot t_i^{-1} \wedge \theta)$. Continuing this procedure which reduces the powers of bigger exponents, we will reach to a formula like $\exists x \left(\bigwedge_i x^n = t_i \wedge \bigwedge_j x^{m_j} \neq s_j \right)$ which is equivalent to $\bigwedge_i t_i = t_0 \wedge \exists x (x^n = t_0 \wedge \bigwedge_j x^{m_j} \neq s_j)$. Whence we are to eliminate the quantifier of the formula

$$(B) \quad \exists x (x^n = t \wedge \bigwedge_{j=1}^{j=l} x^{m_j} \neq s_j).$$

By the equivalence $y^k = a^k \iff \bigvee_{i < n} y = b(\omega_n)^i$ which holds in \mathbb{C} we have the following equivalence in \mathbb{C} : $x^k \neq b \iff x^{kn} \neq b^n \vee \bigvee_{0 < i < n} x^k = b(\omega_n)^i$. For every $j = 1, \dots, l$ and $k = 1, \dots, n$ let the formula $\alpha_{j,k}$ be $x^{m_j} = s_j(\omega_n)^k$ when $k < n$ and for $k = n$ let $\alpha_{j,k} = \alpha_{j,n}$ be the formula $x^{m_j} \neq (s_j)^n$. Then the formula $\bigwedge_{j=1}^{j=l} x^{m_j} \neq s_j$ is equivalent to $\bigwedge_{j=1}^{j=l} \bigvee_{k=1}^{k=n} \alpha_{j,k}$ which is (by Propositional Logic) equivalent to $\bigvee_{f: l \rightarrow n} \bigwedge_{j=1}^{j=l} \alpha_{j,f(j)}$ where $f: l \rightarrow n$ denotes a function from $\{1, \dots, l\}$ to $\{1, \dots, n\}$. So the formula (B) is equivalent to $\exists x (x^n = t \wedge \bigvee_{f: l \rightarrow n} \bigwedge_{j=1}^{j=l} \alpha_{j,f(j)})$ or $\bigvee_{f: l \rightarrow n} \exists x (x^n = t \wedge \bigwedge_{j=1}^{j=l} \alpha_{j,f(j)})$. Hence, we need to eliminate the quantifier of the formula $\exists x (x^n = t \wedge \bigwedge_{j: f(j)=n} (x^n)^{m_j} \neq (s_j)^n \wedge \bigwedge_{j: f(j) \neq n} x^{m_j} = s_j(\omega_n)^{f(j)})$. This formula is in turn equivalent to

$$\bigwedge_{j: f(j)=n} t^{m_j} \neq (s_j)^n \wedge \exists x (x^n = t \wedge \bigwedge_{j: f(j) \neq n} x^{m_j} = s_j(\omega_n)^{f(j)}).$$

So, it suffices to eliminate the quantifier of the formulas in the form

$$(C) \quad \exists x \left(\bigwedge_i x^{k_i} = u_i \right).$$

By the very same procedure that we got to the formula (B) from the formula (A) we can see that the formula (C) is equivalent to a formula of the form $\exists x (x^k = u)$, and finally this is equivalent to the quantifier-free formula $0 = 0$ in $\langle \mathbb{C}, \cdot \rangle$. \square

3 Conclusions

By a theorem of Tarski the theory of the structure $\langle \mathbb{C}, +, \cdot \rangle$ is decidable, whence so is the theory of $\langle \mathbb{C}, \cdot \rangle$. We showed a direct proof of this fact in Theorem 10. Also Tarski showed the decidability of the theory of the structure $\langle \mathbb{R}, +, \cdot \rangle$, and so the theory of the structure $\langle \mathbb{R}, \cdot \rangle$ is decidable too. We also presented a direct proof of this fact in Theorem 9. The references [5, 2, 1] contain some beautiful proofs of the above theorems of Tarski.

By Gödel's incompleteness theorem the theory of the structure $\langle \mathbb{N}, +, \cdot \rangle$ is not decidable (see e.g. [4] or [8] for a proof), and by the four square theorem of Lagrange the set \mathbb{N} is definable in $\langle \mathbb{Z}, +, \cdot \rangle$: for every $n \in \mathbb{Z}$, $n \in \mathbb{N} \iff \exists \alpha, \beta, \gamma, \delta (n = \alpha^2 + \beta^2 + \gamma^2 + \delta^2)$. Whence, the theory of the structure $\langle \mathbb{Z}, +, \cdot \rangle$ is not decidable as well. Also, the theory of the structure $\langle \mathbb{Q}, +, \cdot \rangle$ is not decidable, since \mathbb{Z} is definable in it (see [7]). However, the multiplicative theory of natural and integer numbers, the theories of $\langle \mathbb{N}, \cdot \rangle$ and $\langle \mathbb{Z}, \cdot \rangle$, are decidable (see [3, 8]). The decidability or undecidability of the theory of $\langle \mathbb{Q}, \cdot \rangle$ had remained open in the literature. In Theorem 4 we showed the decidability of the theory of $\langle \mathbb{Q}^+, \cdot \rangle$, and by the technique of the proof of Theorem 9 one can show the decidability of the theory of $\langle \mathbb{Q}, \cdot \rangle$ (Corollary 6).

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